Coxeter Generators and Degenerate Affine Hecke Algebras

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June 27, 2018

Abstract

We will consider the spectra of the YJM elements, defining the set $\mathrm{Spec}(n)$ and outlining how we will use this set to construct the representations of S_n. We will then use the Coxeter Generators of S_n along with their relations to the YJM elements to motivate the definition of the Degenerate Affine Hecke Algebra $H(2)$. By studying the representations of $H(2)$, we will gain valuable information on the elements of $\mathrm{Spec}(n)$, which will bring us closer to constructing the representations of S_n.

1 Review of Prior Theorems/Definitions

The following have all been proved or defined in prior talks; see Micah or David's notes for a more detailed discussion.

Definition 1. The *Gelfand-Tsetlin Algebra* $GZ(n)$ is the algebra generated by the centers Z_i of $\mathbb{C}[S_i] \subset \mathbb{C}[S_n]$ for $1 \leq i \leq n$:

$$GZ(n) = \langle Z_1, \dots Z_n \rangle$$

Definition 2. The *Young-Jucys-Murphy* elements X_i, or *YJM elements*, are elements of $\mathbb{C}[S_i]$:

$$X_i = (1i) + (2i) + \cdots + (i-1, i)$$

In particular, $X_1 = 0$.

Theorem 1. $Z_n \subset \langle Z_{n-1}, X_n \rangle$

Corollary 1. *The YJM-elements are a basis for the Gelfand-Tsetlin algebra, i.e.*

$$GZ(n) = \langle X_1, \dots, X_n \rangle$$

Proof. Proof by induction. The base case is easy: $GZ(2) = \mathbb{C}[S_2] = \langle X_1, X_2 \rangle = \mathbb{C}^2$. Therefore we must use $GZ(n-1) = \langle X_1, \dots, X_{n-1} \rangle$ to prove $GZ(n) = \langle X_1, \dots, X_n \rangle$. Equivalently, we may show $GZ(n) = \langle GZ(n-1), X_n \rangle$, which we will show by demonstrating both inclusions.

$GZ(n) \supset \langle GZ(n-1), X_n \rangle$ is obvious since $X_n \in GZ(n)$ and $GZ(n-1) \subset GZ(n)$.

For $GZ(n) \subset \langle GZ(n-1), X_n \rangle$, note $GZ(n) = \langle GZ(n-1), Z_n \rangle$ by definition, so it suffices to show $Z_n \subset \langle GZ(n-1), X_n \rangle$. This follows from the above theorem:

$$Z_n \subset \langle Z_{n-1}, X_n \rangle \subset \langle G_{n-1}, X_n \rangle$$

□

Theorem 2. *The branching graph of S_n is simple.*

Corollary 2. *The algebra $GZ(n)$ is a maximal commutative subalgebra of $\mathbb{C}[S_n]$. Thus the Gelfand-Tsetlin basis is determined in each irreducible representation of S_n up to scalar factors.*

This corollary allows us to make a definition:

Definition 3. The *Young basis* $\mathcal{Y}$ is the union of all Gelfand-Tsetlin bases of all irreducible representations of S_n:

$$\mathcal{Y} = \coprod_{\lambda \in \widehat{S_n}} \{v_T\}_\lambda$$

Where we've denoted the Gelfand-Tsetlin basis in the irreducible representation λ by $\{v_T\}_\lambda$.

This last proposition has not explicitly been mentioned before, but will be useful for a later proof:

Proposition 1. *Let $T = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n$. Then*

$$\mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$$

Proof. Firstly note that for any $v \in V^\lambda$, with V^λ an irreducible representation of G, we have $\mathbb{C}[G] \cdot v = V^\lambda$ as $\mathbb{C}[G] \cdot v$ is a nonempty subrepresentation of V^λ. Since $v_T \in V^{\lambda_i}$ the proposition follows. □

2 Weights and Spec

Recall from earlier (David's talk) that the Gelfand-Tsetlin algebra is the algebra of all operators diagonal in the Gelfand-Tsetlin basis. This holds in any basis, so in particular it holds for the YJM-elements. Therefore we may make the following definition:

Definition 4. For an element $v \in \mathcal{Y}$, the *weight* of v denoted $\alpha(v)$ is given by

$$\alpha(v) = (a_1, \ldots, a_n) \qquad a_i \in \mathbb{C}$$

Where $X_i v = a_i v$ for all i.

As David proved earlier, an element $v \in \mathcal{Y}$ is determined solely by the eigenvalues of the elements of $GZ(n)$ acting of v. Therefore $v \to \alpha(v)$ is a one-to-one correspondence, the inverse of which we will refer to by $\alpha \to v_\alpha$.

We can put all of the weights together into a set:

$$\operatorname{Spec}(n) = \{\alpha(v) \mid v \in \mathcal{Y}\}$$

By the above, we see $\operatorname{Spec}(n)$ is in bijection with the Young basis, so we have

$$|\operatorname{Spec}(n)| = |\mathcal{Y}| = \sum_{\lambda \in \widehat{S_n}} \dim V^\lambda$$

Furthermore we can extend this bijection to paths T in the branching graph; we will denote this bijection by

$$T \to \alpha(T), \qquad \alpha \to T_\alpha$$

There is also a natural equivalence relation on $\operatorname{Spec}(n)$. Denoting the relation by $\sim$, we have $\alpha \sim \alpha'$ if and only if v_α and $v_{\alpha'}$ lie within the same irreducible representation. Or, equivalently, if T_α and $T_{\alpha'}$ end in the same position. It is easy to see that $\operatorname{Spec}(n)/\sim$ is in bijection with the set of irreducible representations $\widehat{S_n}$, or otherwise stated:

$$|\operatorname{Spec}(n)/\sim| = |\widehat{S_n}|$$

The procedure for the remainder of the proof will be as follows:

1. Describe the set $\operatorname{Spec}(n)$
2. Describe the equivalence relation $\sim$
3. Use the above to compute matrix elements in the Young basis (i.e., compute the irreducible representations)
4. Compute the characters of the irreducible representations

In order to begin the first of these items, we will begin with the study of Coxeter Generators

3 Coxeter Generators

Definition 5. The *Coxeter generators* $\{s_i\}$ of S_n are given by transpositions of adjacent elements, i.e.

$$s_i = (i,\, i+1) \qquad 1 \leq i < n$$

An important feature of note here is that these generators mostly commute with each other, except for adjacent elements:

$$s_i s_j = s_j s_i, \qquad i \neq j \pm 1 \tag{3.1}$$

This property has been termed *locality* (a term taken from physics), and leads to the following proposition:

Proposition 2. *Let $T = \lambda_0 \nearrow \cdots \nearrow \lambda_n$. For any $1 \leq k < n$, we can write $s_k \cdot v_T$ as a linear combination of vectors $v_{T'}$, $T' = \lambda'_0 \nearrow \cdots \nearrow \lambda'_n$ satisfying*

$$\lambda'_i = \lambda_i \qquad i \neq k$$

This proposition captures the idea that the Coxeter generators act locally on the branching graph, with s_k only affecting the branching in s_k.

Proof. Suppose $i > k$. Then by Proposition 1 we have

$$\mathbb{C}[S_i] \cdot (s_k \cdot v_T) = (\mathbb{C}[S_i]s_k) \cdot v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$$

Which shows that $s_k \cdot v_T$ contains only vectors $v_{T'}$ which contain the branch λ_i.

If $i < k$ then s_k commutes with everything in S_i (see (3.1)), so we have

$$\mathbb{C}[S_i]s_k \cdot v_T = s_k \cdot \mathbb{C}[S_i] \cdot v_T = s_k \cdot V^{\lambda_i}$$

But also since s_k commutes with S_i, left multiplication by S_i is an S_i-morphism, so $s_k \cdot V^{\lambda_i} = V^{\lambda_i}$ and the proof follows as above. $\square$

4 The Degenerate Affine Hecke Algebra $H(2)$

4.1 Motivation and definitions

In order to use the Coxeter generators to study the representations of S_n, we wish to study their relation to another object we've been using: the YJM elements. Due to the locality of the Coxeter generators, this has a particularly simple form in most cases:

$$s_i X_j = X_j s_i, \qquad j \neq i, i+1$$

Meanwhile, for the edge cases we have a different relation, which can be easily verified:

$$s_i X_i + 1 = X_{i+1} s_i$$

This last relation will be important for us. Since this is a local relation, we can use this relation to study $\mathbb{C}[S_n]$ only at the point where we transition from $\mathbb{C}[S_i]$ to $\mathbb{C}[S_{i+1}]$, i.e. at the ith level of the branching graph. Therefore we consider the algebra generated by this relation:

Definition 6. The algebra $H(2)$ is defined as the algebra generated by elements Y_1, Y_2, and s with the following relations:

$$s^2 = 1, \qquad Y_1 Y_2 = Y_2 Y_1 \qquad sY_1 + 1 = Y_2 s$$

Note this is the algebra generated by s_i, X_i, and X_{i+1}, except phrased in a more abstract sense. From the above, the next proposition should be obvious:

Proposition 3. *The algebra $\mathbb{C}[S_n]$ is generated by $\mathbb{C}[S_{n-1}]$ and $H(2)$, where we recognize $H(2)$ as the subalgebra generated by*

$$Y_1 = X_{n-1} \qquad Y_2 = X_n \qquad s = s_{n-1}$$

□

Note that $\mathbb{C}[S_{n-1}]$ and s_n alone generate $\mathbb{C}[S_n]$. However, by adding these additional generators we allows ourselves to use the study of the irreducible representations of $H(2)$ to help build up the representation theory of S_n. In particular, we may restrict any representation of S_n to a representation of $H(2)$, so if we may find conditions on the representations of $H(2)$ then these will lead to conditions on the representations of S_n.

Remark 1. As the notation suggests, what we have defined as $H(2)$ is only one of an infinite chain of algebras. The nth such algebra, denoted $H(n)$, is generated by elements $Y_1, \dots Y_n$ and $s_1, \dots, s_{n-1}$ with relations

$$Y_iY_j = Y_jY_i \qquad s_i^2 = 1$$
$$Y_is_j = s_jY_i \qquad j \neq i, i+1$$
$$s_iY_i + 1 = Y_{i+1}s_i$$

We will only be using the algebra $H(2)$.

4.2 Irreducible representations of $H(2)$

In order to study the irreducible representations of $H(2)$, we first note that Y_1 and Y_2 commute by definition; therefore they have a common eigenbasis and we can find a vector v in any representation such that

$$Y_1v = av$$
$$Y_2v = bv$$

Now consider the span of v and $s\cdot v$, where $s \in H(2)$. It is easy to check from the relations above that this subspace is closed under the action of $H(2)$; therefore $\operatorname{span}\{v, sv\}$ forms a subrepresentation and every irreducible representation of $H(2)$ must be at most 2-dimensional.

We should note, before proceeding, that the particular case we are interested in is $Y_1 = X_i$, $Y_2 = X_{i+1}$, and $s = s_i$. Under this correspondence, if v is an element of the Young basis and $\alpha(v) = (a_1, \dots, a_n)$ is its weight, we have

$$a = a_i$$
$$b = a_{i+1}$$

This correspondence will allow us to conclude restraints on the elements of $\operatorname{Spec}(n)$ from the irreducible representations of $H(2)$.

Working in the basis $\{v, sv\}$, we can compute matrix elements for $H(2)$, e.g.,

$$Y_2sv = (sY_1 + 1)v = a(sv) + v$$

This allows us to read off the second column of the matrix representing Y_2. Repeating for every required matrix element, we have the following matrices:

$$Y_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix} \qquad Y_2 = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix} \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can compute the eigenvectors of these matrices. Both Y_1 and Y_2 share have a common eigenbasis, and v is one such eigenvector by definition. We may also find another eigenvector w given by:

$$w = \begin{pmatrix} 1 \\ b-a \end{pmatrix} = v + (b-a)sv$$

Note that if $a = b$, then this second eigenvector is degenerate, i.e. $v = w$. Therefore Y_i are diagonalizable if and only if $a \neq b$. Since the YJM elements are always diagonalizable (in particular, they are diagonal in the Young basis), when we set $Y_1 = X_i$ and $Y_2 = X_{i+1}$ this rules out the possibility of a common eigenvalue of X_i and X_{i+1} on the same eigenvector. For $v \in \mathcal{Y}$, this gives us the restriction that if $\alpha(v) = (a_1, \ldots, a_n)$, then $a_i \neq a_{i+1}$ for any i.

At this point, we split off into two cases: $b = a \pm 1$ and $b \neq a \pm 1$.

If $b = a \pm 1$, then we have $w = v \pm sv$. Applying s to w, we have

$$sw = s(v \pm sv) = sv \pm v = \pm w$$

Thus we see w is a common eigenvector of every element of $H(2)$, which means that $\text{span}\{w\}$ is a subrepresentation. We can also obtain a converse to this fact: suppose that u is a vector such that v and sv are proportional. Then in order to have $s^2 = 1$ we must have

$$sv = \pm v$$

Thus we may compute

$$bv = Y_2 v = \pm Y_2 sv = \pm(sY_1 + 1)v = (a \pm 1)v$$

$$\implies b = a \pm 1$$

In terms of $\text{Spec}(n)$, we make the substitution $a = a_i$ and $b = a_{i+1}$. This tell us that if (and only if) $a_{i+1} = a_i \pm 1$, then $s_i v = \pm v$ for $v_\alpha \in \mathcal{Y}$.

Meanwhile, if $a_{i+1} \neq a_i \pm 1$, then the 2-dimensional representation defined above is irreducible. This means v and w, taken to be elements of the Young basis, fall within the same irreducible representation, since we can act $s + b - a = s_i + a_{i+1} - a_i$ on v to obtain w. Furthermore we can deduce the action of $X_i = Y_1$ and $X_{i+1} = Y_2$ purely from a_i and a_{i+1}. Working in the basis $\{v, w'\}$, where $w' = (a_{i+1} - a_i)^{-1} w$ (we choose this normalization for properties which go beyond the scope of this talk), we can compute the matrix elements:

$$X_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix} \qquad X_{i+1} = \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix} \qquad s_i = \begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{pmatrix} \tag{4.1}$$

4.2.1 Summary

We can summarize all of this in the following proposition:

Proposition 4. *Let*

$$\alpha = (a_1, \ldots, a_i, a_{i+1}, \ldots a_n) \in \text{Spec}(n)$$

Then

1. $a_i \neq a_{i+1}$ *for all* i
2. *if* $a_{i+1} = a_i \pm 1$, *then* $s_i \cdot v_\alpha = \pm v_\alpha$
3. *if* $a_{i+1} \neq a_i \pm 1$, *then*

$$\alpha' = s_i \cdot \alpha = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \text{Spec}(n)$$

and $\alpha' \sim \alpha$ *(i.e.* v_α *and* $v_{\alpha'}$ *lie in the same irreducible representations). Additionally, we have*

$$v_{\alpha'} = \left(s_i - \frac{1}{a_{i+1} - a_i}\right) v_\alpha$$

and the action of s_i, X_i, *and* X_{i+1} *on the basis* $\{v_\alpha, v_{\alpha'}\}$ *is given by the matrices in (4.1)*

Corollary 3. *Given $\alpha \in \mathrm{Spec}(n)$ as above, $a_i \in \mathbb{Z}$ for all i.*

Proof. Suppose not; then there exists some first $i \geq 1$ such that $a_i = a \notin \mathbb{Z}$. Using item 3 in the above proposition,[1] we can apply s_{i-1} to α to obtain $\alpha' \in \mathrm{Spec}(n)$ such that $a_{i-1} = a \notin \mathbb{Z}$. Repeating this procedure, we can move a to the left until $a_1 = a \neq 0$. However, since $X_1 = 0$ we must have $a_1 = 0$, which is a contradiction. □

As we can see, this analysis has greatly restricted which vectors can be in $\mathrm{Spec}(n)$. For example, a_2 must be ± 1, as otherwise we could switch a_2 and a_1 to obtain an element of $\mathrm{Spec}(n)$ with $a_1 \neq 0$.

[1] We can never have $a_i = a_{i-1} \pm 1$ since $a_{i-1} \in \mathbb{Z}$ and $a_i \notin \mathbb{Z}$